CONTACT HEAT TRANSFER IN GRANULAR MATERIAL UNDER VACUUM

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By integration of the Laplace equation, equations have been obtained for the temperature field and the thermal resistance of a sphere with two contact areas. A formula has been derived for determining the contact thermal conductance of granular materials.

In a granular material under vacuum, heat transfer is accomplished by two means: conduction and radiation. In many cases heat transfer by the first means is predominant. This has been examined in references [1–3].

Transfer of heat in a granular material by conduction occurs through the areas of contact between adjoining grains. To calculate heat transfer in this case we first find the temperature field in a sphere with two contact areas (Fig. 1). We shall take as given the radius of the sphere r_0 and of the contact area a. For simplicity, we will assume that the contact spot is not plane, but rather part of the surface of the sphere, an assumption which will not introduce an appreciable error for a. For example, with $a = 0.2$, the deformation of the sphere when a plane contact spot is formed is only $0.02\ r_0$, and the area of the plane spot is less than that of the corresponding section of the spherical surface by 1%. We also consider, as in the case of a plane contact spot, that $\alpha = \sin\varphi$.

The solution of the problem reduces to integration of the Laplace equation in spherical coordinates

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial T}{\partial\vartheta}\right)=0 \quad (1)$$

with boundary conditions

$$\left.\frac{\partial T}{\partial r}\right|_{r=r_0}=\left.\frac{\partial T}{\partial\rho}\right|_{\rho=1}=f(\vartheta)=$$

$$=\begin{cases}-\dfrac{q}{\lambda_T} & \text{for } 0<\vartheta<\varphi\\ 0 & \text{for } \varphi<\vartheta<\pi-\varphi\\ +\dfrac{q}{\lambda_T} & \text{for } \pi-\varphi<\vartheta<\pi\end{cases}. \quad (2)$$

The solution of (1) has the form

$$T=\sum_{k=0}^{\infty}a_k\rho^kP_k(\cos\vartheta)\quad(\rho\leqslant 1), \quad (3)$$

where $P_k(\cos\vartheta)$ are Legendre polynomials of the first kind of k-th order.

The function $f(\vartheta)$ may be expanded in a series of Legendre polynomials,

$$f(\vartheta)=\sum_{k=0}^{\infty}b_kP_k(\cos\vartheta), \quad (4)$$

where

$$b_k=\frac{2k+1}{2}\int_0^{\pi}f(\vartheta)P_k(\cos\vartheta)\sin\vartheta\,d\vartheta. \quad (5)$$

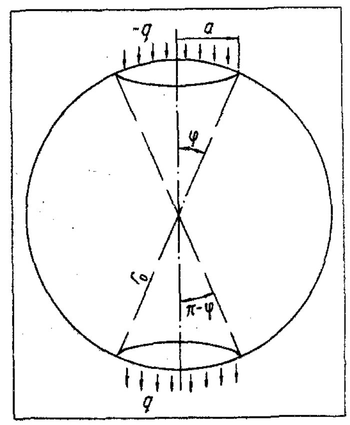

Fig. 1. Heat transfer scheme for a sphere with two contact areas.

After finding the derivative $\partial T/\partial\rho$ from (3), and equating it to $f(\vartheta)$ when $\rho = 1$, we obtain a relation between the coefficients a_k and b_k:

$$a_k=b_k/k. \quad (6)$$

We determine the coefficients b_k from (5):

$$b_k=\frac{2k+1}{2}\left[\int_0^{\varphi}\left(-\frac{q}{\lambda_T}\right)P_k(\cos\vartheta)\sin\vartheta\,d\vartheta+\right.$$

$$\left.+\int_{\pi-\varphi}^{\pi}\frac{q}{\lambda_T}P_k(\cos\vartheta)\sin\vartheta\,d\vartheta\right]=$$

$$=\frac{(2k+1)q}{2\lambda_T}\left[-\int_1^{\cos\varphi}P_k(x)\,dx+\int_{-\cos\varphi}^{-1}P_k(x)\,dx\right]. \quad (7)$$

To calculate the integrals in (7) we use a recurrence relation for the functions $P_k(x)$:

$$(2k+1)P_k(x)=P'_{k+1}(x)-P'_{k-1}(x). \quad (8)$$

Integrating in the range 1 to $\cos\varphi$, we obtain

$$\int_1^{\cos\varphi}P_k(x)\,dx=\frac{1}{2k+1}\left[P_{k+1}(\cos\varphi)-P_{k-1}(\cos\varphi)\right]. \quad (9)$$

The second integral in (7) is equal in value to the first.

Both integrals have identical signs for odd k and opposite signs for even k. Therefore the even coefficients b_k are zero. Finally, we find

$$b_{2n+1} = \frac{q}{\lambda_T}\left[P_{2n}(\cos\varphi) - P_{2n+2}(\cos\varphi)\right], \qquad (10)$$

where

$$n = (k-1)/2 \quad (k = 1, 3, 5, \ldots).$$

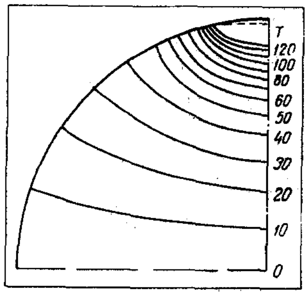

Fig. 2. Temperature field in a sphere with two contact areas, with $\alpha = 0.2$ and $q/\lambda_T = 1000$.

Taking (6) into account, we obtain a formula for the temperature field in the sphere

$$T = \frac{q}{\lambda_T}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[P_{2n}(\cos\varphi) - P_{2n+2}(\cos\varphi)\right]\rho^{2n+1}P_{2n+1}(\cos\vartheta), \qquad (11)$$

or

$$T = \frac{q}{\lambda_T}\left[\frac{3}{2}\alpha^2\rho\cos\vartheta + \frac{7}{16}(2\alpha^2 - 5\alpha^4)\rho^3(5\cos^3\vartheta - 3\cos\vartheta) + \frac{1}{128}(88\alpha^2 - 308\alpha^4 + 231\alpha^6)\rho^5(63\cos^5\vartheta - 70\cos^3\vartheta + 15\cos\vartheta) + \ldots\right]. \qquad (12)$$

It is more convenient to perform the calculations from (11) using tables of Legendre polynomials (see, for example, [4]). The convergence of the series in (11) deteriorates as $\rho \to 1$ and $\vartheta < \varphi$.

As an example we will find the temperature field of a sphere with two contact areas, for which $\alpha = 0.2$. Then (11) takes the form

$$T = \frac{q}{\lambda_T}\left[0.0600\rho P_1(\cos\vartheta) + 0.0421\rho^3 P_3(\cos\vartheta) + 0.0378\rho^5 P_5(\cos\vartheta) + 0.0319\rho^7 P_7(\cos\vartheta) + 0.0258\rho^9 P_9(\cos\vartheta) + \ldots\right].$$

The temperature field in the sphere, constructed from this relation, is shown in Fig. 2, making the assumption that $q/\lambda_T = 1000$.

We will determine the thermal resistance of a sphere with two contact areas. For simplicity of calculation we assume that the temperature of the contact area is equal to the arithmetic mean of the temperatures at the center and at the periphery of the area, an assumption that will not introduce appreciable error into the calculation. As shown in [5], in the case of contact heat transfer, the error is comparatively small even when the boundary condition of constant heat flux is replaced by the condition of constant temperature over the whole of the contact area:

$$2T_m = T_{\substack{\vartheta=0\\ \rho=1}} + T_{\substack{\vartheta=\varphi\\ \rho=1}} = \frac{q}{\lambda_T}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[P_{2n}(\cos\varphi) - P_{2n+2}(\cos\varphi)\right]\left[1 + P_{2n+1}(\cos\varphi)\right]. \qquad (13)$$

The temperature at the equatorial section of the sphere is zero. Therefore,

$$R_s = \Delta T/Q_1 = 2T_m/q\pi\sin^2\varphi,$$

and, in final form,

$$R_s = \frac{1}{\pi a\lambda_T\sin\varphi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\times\left[P_{2n}(\cos\varphi) - P_{2n+2}(\cos\varphi)\right]\left[1 + P_{2n+1}(\cos\varphi)\right]. \qquad (14)$$

The thermal resistance of a circular contact area on a semiinfinite body is equal to [5]

$$R^T_\infty = 1/4a\lambda_T \qquad (15)$$

under the condition of constant temperature on the area, and

$$R^q_\infty = 8/3\pi^2 a\lambda_T \qquad (15')$$

under the condition of constant heat flux at the area.

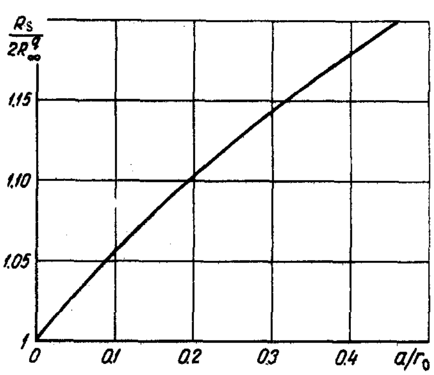

Fig. 3. Dependence of the ratio of thermal resistance of a sphere and a semiinfinite body with a contact area on the radius of the area.

The values of R^T_∞ and R^q_∞ differ by 8%. The ratio

$$\frac{R_s}{2R^q_\infty} = \frac{3\pi}{16\alpha}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[P_{2n}(\cos\varphi) - P_{2n+2}(\cos\varphi)\right]\left[1 + P_{2n+1}(\cos\varphi)\right] \qquad (16)$$

depends only on the relation $\alpha = a/r_0 = \sin\varphi$ and indicates an increase of contact thermal resistance on transition from a semiinfinite body to a sphere.

The relation given by (16) is shown in Fig. 3. The thermal resistance of a sphere with two contact spots is somewhat greater than the resistance of the two contact spots on a semiinfinite body, and approaches the value of the latter as a/r_0 decreases.

In [3] the assumption was made that the thermal resistance of a spherical particle at the point of contact is equal to the thermal resistance of a semiinfinite body at the point of contact. Calculation of this resistance was carried out incorrectly, and the resulting erroneous value obtained was

$$R = 1/2\pi a \lambda_{\text{т}},$$

which is $\pi/2$ times less than R^{T}_{∞}, calculated from (15).

We now turn to the determination of the thermal conductivity of the granular material. We assume that the grains are spherical. The thermal conductivity of the granular material is

$$\lambda = Qh/s\Delta T. \qquad (17)$$

Each sphere in a bed of spherical particles touches N neighboring spheres, and therefore N/3 contacts are associated with the direction of each of three mutually perpendicular axes, and N/6 contacts are associated with each of the two opposite directions along one of the axes. In fact, the sphere cannot make contact in any direction with more than one sphere. The physical meaning of the above statement is that the numerical value of the vector heat flux through one sphere in the direction of the temperature gradient is

$$|\vec{Q}| = Q_1 N/6. \qquad (18)$$

The corresponding value of the quantity h is $2r_0$. The bed area, taken over one grain, is related to the other bed parameters by the relation

$$m = 1 - 4\pi r_0^3/3hs, \qquad (19)$$

whence

$$s = 2\pi r_0^2/3(1-m). \qquad (20)$$

The theoretical values of N are known only for certain types of regular packing, given in the table. It may be supposed that in real random packing the particles are located uniformly throughout the volume and the number of contacts is determined by the porosity of the packing. The dependence of the number of contacts on the volume occupied by the spheres may be expressed, for m > 0.3, by the empirical relation

$$N = 11.6(1-m), \qquad (21)$$

valid for regular packing with coordination numbers 4, 6, and 8.

When a pressure acts on a bed of spherical particles, contact areas are formed at their points of contact, the radius of the areas being given by the Hertz relation

$$a = \sqrt[3]{\frac{3}{4}\,\frac{1-\mu^2}{E}\,P_1 r_0}. \qquad (22)$$

Characteristics of Regularly Packed Spheres

Type of packing	Coordination number	Porosity, %
Densest (hexagonal close-packed, face-centered cubic)	12	25.95
Body-centered cubic	8	31.98
Simple cubic	6	47.64
Structure of diamond and ice	4	65.99

We will find the mean value of P_1, taking account of (20) and (21):

$$P_1 = \frac{ps}{N/6} =$$

$$= p\,\frac{2\pi r_0^2}{3(1-m)}\,\frac{6}{11.6(1-m)} = \frac{0.345}{(1-m)^2}\,p\pi r_0^2. \qquad (23)$$

Substituting the value of P_1 into (22), we obtain

$$a = 0.93 r_0 \sqrt[3]{(1-\mu^2)\,p/E\,(1-m)^2}. \qquad (24)$$

The heat flux through one sphere is

$$Q = \Delta T N/R_s \cdot 6. \qquad (25)$$

Calculations according to (24) show that for specific load values up to 0.1 MN/m^2 for ordinary granular materials with porosity 40–50%, the ratio a/r_0 does not exceed 0.02, while for insulating materials with porosity up to 95%, it is close to 0.1. It may be seen from Fig. 3 that in these cases, to an accuracy sufficient for technical purposes, the quantity R_s in (25) may be replaced by $2R_\infty$. When the specific load is increased, formula (14) or Fig. 3 must be used to find R_s.

Substituting (15'), (20), (24) and (25) into (17), we finally find

$$\lambda = 3.12\,(1-m)^{4/3}\,\lambda_{\text{т}}\,p^{1/3}/E^{1/3}. \qquad (26)$$

Reference [6] gave the following experimental values of thermal conductivity of sand with grain size less than 0.2 mm and of glass beads of diameter 4.8 mm, under vacuum and at mean temperature of 338° K:

	Thermal conductivity, mW/m · degree		
	No load (λ_0)	With load 0.1 MN/m^2 (λ_p)	$\lambda_p - \lambda_0$
Sand	3.64	36.0	32.4
Glass beads	11.0	40.0	29.0

The characteristics of the materials were not given in the paper. Assuming for sand the approximate values m = 0.4; $E = 5 \cdot 10^{10}$ N/m^2 and $\lambda_T = 3$ W/m · degree, we find from (26) that $\lambda = 59$ mW/m · degree.

For glass beads with λ_T = 1.0 W/m · degree and the same values of the other parameters, we obtain λ = = 19.7 mW/m · degree. The larger discrepancy between the calculated and experimental values in the first case may be due to deviation of the shape of the sand particles from spherical.

In [7] a determination was made of the thermal conductivity under load for silica aerogel of density 120 kg/m^3 under vacuum and at a mean temperature of 307° K. A comparison is made below of the experimental data and the calculated values (in calculating the variation of thermal conductivity with load, an initial value of λ at 0.037 MN/m^2 was assumed):

Load, MN/m^2	0.0028	0.037	0.105
λ_{exp}, mW/m · degree	5.31	5.86	6.28
λ_{calc}, mW/m · degree	5.45	5.86	6.15

For the aerogel λ_T was taken to be equal to λ for amorphous silica, i.e., 1.3 mW/m · degree. The porosity of the aerogel m = 0.95 was found from its density.

Thus, relation (26) is in satisfactory agreement with the experimental data, and may be used for calculation of contact thermal conductivity of granular materials under vacuum.

NOTATION:

E—modulus of elasticity; h—height of bed of grains corresponding to temperature difference ΔT; m—porosity; p—specific compressive force on granular material in direction of temperature gradient; P_1—force acting on one contact area; Q_1—heat flux through contact area; q—specific heat flux; R_S—thermal resistance of a sphere with two contact areas; s—area in grain bed corresponding to a single grain; $\alpha = a/r_0$—dimension less radius of a contact area; λ_T—thermal conductivity of sphere material; μ—Poisson's ratio; $\rho = r/r_0$—dimensionless sphere radius.

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